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From chaotic to disordered systems—a periodic orbit approach

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Abstract. We apply periodic orbit theory to a quantum billiard on a torus with a variable number (N) of randomly distributed circular scatterers. Provided the scatterers are much smaller than the wavelength they may be regarded as sources of pure s-wave diffraction. The relevant part of the spectral determinant is due only to diffractive periodic orbits. We formulate this diffractive zeta function in terms of an $N \times N$ transfer matrix, which is transformed to real form. The determinant is shown to reproduce the full density of states for generic configurations if $N \geq 4$. The zeros of the determinant are computed numerically. We study the statistics exhibited by these spectra. The numerical results suggest that the spectra tend to GOE statistics as the number of scatterers increases for typical members of the ensemble. A peculiar situation arises for configurations with four scatterers and kR tuned to $kR = y_{0,1} \approx 0.899$, where the statistics appears to be perfectly Poissonian.

1. Introduction

Universal level statistics of classically chaotic systems is an asymptotic (i.e. semiclassical) property of a spectrum. For homogeneous systems, such as billiards, it appears in the high-energy limit of the spectrum. Usually there is a pre-asymptotic regime where the statistics reflects characteristics of classical or quantum origin specific to the system.

Studies of non-universal level statistics, and the approach to universal level statistics, as proposed by random matrix theories, have proceeded mainly along the following three lines.

- (1) Via classical diffusion and inhibition of quantum diffusion due to dynamical localization [1].
- (2) Via nonlinear σ models and supersymmetric techniques [2].
- (3) Via periodic orbit theories.

This first approach has been studied for periodically driven systems but generalization to billiards has recently been suggested [3]. The second approach has shown dynamical progress during recent years. In this paper we will pursue the third approach.

Admittedly, periodic orbit theories based on the semiclassical trace formula [4] have not been very successful for studies of spectral statistics. One exception is Berry's theory of long-range spectral correlations [5], or equivalently, the small τ limit of the form factor $K(\tau)$, which can be related to periodic orbits. This limit can be related to the large τ limit by assuming that the spectra are real [6]. Under the *diagonal approximation* these result can be extended to intermediate values of τ but the conditions under which this approximation is valid are obscure [7].

The possibility of tracing level statistics to classical dynamics very much hinges on the ability of the trace formula to produce real eigenvalues. In this introduction, we will discuss various problems of the trace formula in this respect. After that we will suggest a role for periodic orbit theories where none of these problems are recognized.

To simplify the discussion of periodic orbit theories below we will confine ourselves to dispersive billiards consisting of one or several discs, all with radius R , inside a rectangle or a torus. We thus limit the number of relevant length scales to essentially one. One obvious condition for the trace formula to be applicable is $kR \gg 1$, where $k = \sqrt{2mE}/\hbar$ is the wavenumber[†]. Non-universal effects are expected to be pronounced for intermediate values of kR , and are thus to be sought in the region where the validity of the trace formula is dubious from the very beginning.

Diffraction presumably plays a much more important role in a bound system than in an open one, and the Gutzwiller formula has to be amended by sums over diffractive periodic orbits [8, 9]. However, it is not obvious how much the inclusion of diffractive orbits will improve the situation. The *geometric theory of diffraction* has been very successful for open systems, where it provides small corrections, but its applicability is questionable in the *penumbra* of disc scattering [10, 11]. The basic problem is the interference between trajectories being scattered in an extreme forwards direction and trajectories sneaking close to the disc. Their respective saddle points are not separated enough and individual eigenstates cannot be resolved by the Berry–Keating technique [12]. This should hardly come as a surprise. The failure to predict individual eigenvalues in the semiclassical limit has been expected from the very beginnings of the trace formula. However, this pessimism has lessened since the successes of the Berry–Keating scheme [13, 14].

The situation may be better for systems without neutral orbits. But even if a model system may be found for which the trace formula is accurate enough, one still faces more practical problems. In numerical studies one is plagued by the exponential growth of the number of periodic orbits. Approaching the semiclassical limit of the spectrum is a truly painstaking experience. In analytical studies one has to control the asymptotics of the periodic orbits, beyond the flow conservation sum rule [15].

The harsh moral of this discussion is that it is not an easy task to pursue periodic theories as a means to study the emergence of universality as $kR \rightarrow \infty$. We have discussed two sources of problems: the first concerned the validity of the trace formula and the second the exponential proliferation of human and computer labour to actually compute things.

The situation turns out to be very different in the opposite limit ($kR \ll 1$). The discs can now be considered as sources of pure s-wave diffraction. The unstable orbits are replaced by purely diffractive orbits. The exponential explosion of the periodic orbit can be tamed, because the zeta function can, as we will see, be written in terms of a finite transfer matrix. So, it is within reach to study level statistics within the framework of periodic orbit theories.

In [16] we studied the small kR limit for the one-disc case. This limit is not only much easier to deal with, it is also much richer in behaviour. In particular, the A_1 subspace exhibits a wide range of level statistics in the diffractive region which, due to symmetry effects, extends up to $kR \approx 4$. For very small values of kR the statistics approaches the Poissonian; a similar effect may be observed at $kR \approx 2.40$ but corrections to the diffraction approximation probably distort the effect slightly. When $kR \approx 0.899$ the statistics is very close to GOE. It should approach GOE properly first in the limit $kR \rightarrow \infty$. One of the questions we will address is whether GOE statistics can be achieved by keeping kR small, and increasing the number of scatterers, and distribute them randomly over a torus. We will thus enter the realm of disordered systems.

[†] From now on we will use units such that $m = \hbar = 1$.

Such a *disordered Lorentz* gas has been studied in [17] in the the Gutzwiller limit $kR \gg 1$.

We have chosen to study spectral statistics for *individual* members of our disordered ensembles, for various values of the parameters (N and kR). Any study of non-universality and approaches to universality for a single chaotic system (utilizing self-averages of the spectrum) will suffer from finiteness of the sample. Some non-universal effects, like fine wiggles on the form factor proposed by [18, 2] may not relevant when applied to a single system [19], but this is not the kind of effect we will be looking at.

The basic motivation of this paper is conceptual rather than physical, although our studies do have bearing on impurity scattering in mesoscopic devices. The system considered in this paper has obvious similarities with antidot lattices [20]. However, present-day antidot lattices are modelled by rather smooth potentials and they lie in the intermediate region $kR \sim 2\pi$ (with R suitably defined), a region where semiclassical methods are not very attractive.

The outline is as follows. The eigenvalues are recognized as the zeros of the *spectral determinant*, which we will derive within the geometric theory of diffraction. We will focus on the diffractive determinant (or zeta function), associated with periodic orbits with at least one scattering on a diffractive object. This is formulated in section 2.1. The determinant exhibits poles on the real axis and will diverge unless it is resummed, which is done in section 2.2. In section 2.3 we discuss some numerical issues. In section 3 we derive the mean level density of zeros of the diffractive determinant. In particular we show that the full density of states is reproduced by the determinant first if $N \geq 4$. In section 4 we compute spectra numerically and study their statistics.

2. The diffractive determinant

2.1. Derivation of the diffractive determinant

In the *geometric theory of diffraction* [8, 9, 21] the spectral determinant is divided into a product

$$\Delta(E) = \Delta_0(E) \cdot \Delta_G(E) \cdot \Delta_D(E) \quad (1)$$

where $\Delta_0(E)$ corresponds to the mean level density; the geometric part, $\Delta_G(E)$, is the Gutzwiller–Voros zeta function, possibly amended with the neutral orbits. We will be solely interested in the diffractive determinant $\Delta_D(E)$. It was derived for a non-diffractive system supplied with N small discs in [16]. The result will now be reviewed.

We thus assume the presence of N small diffractive objects located at \mathbf{r}_j , where $1 \leq j \leq N$ whose diffraction constants $d_j(E)$ do not depend on the scattering angle. We introduce symbolic dynamics by enumerating the disc from 1 to N . The alphabet is now $\{j; 1 \leq j \leq N\}$. The set Ω_D is defined as the set of all primitive periodic sequences of symbols taken from this alphabet.

The diffractive determinant (or zeta function) is now given by

$$\Delta_D(E) = \prod_{p \in \Omega_D} (1 - t_p). \quad (2)$$

The weight t_p is given by

$$t_p = \prod_{i=1}^{n_p} d_{k_i} G_G(\mathbf{r}_{k_{i-1}}, \mathbf{r}_{k_i}, E) \quad (3)$$

where $p = \overline{k_1 k_2 \dots k_{n_p}}$, and $k_0 = k_n$. $G_G(\mathbf{r}, \mathbf{r}', E)$ is the geometric Green function, that is the Green function associated with the original system, before any diffractive objects have been inserted. We will choose, as this underlying system, a rectangle with sides a and b supplied with periodic boundary conditions.

The N diffractive objects will be small discs, all with the same radius R . The diffraction constant is then given by [24]

$$d(E) = -4i \frac{J_0(kR)}{H_0^{(1)}(kR)} \quad (4)$$

provided that kR is small.

The geometric Green function $G_G(\mathbf{r}, \mathbf{r}', E)$ can be expressed in terms of the free Green function in two dimensions

$$G_0(\mathbf{r}, \mathbf{r}', E) \equiv G_0(\mathbf{r}' - \mathbf{r}, E) = -\frac{i}{4} H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|) \quad (5)$$

by the method of images

$$G_G(\mathbf{r}, E) = \sum_{\rho=(ma, nb)} G_0(\mathbf{r} + \rho, E) \quad (6)$$

where the sum runs over all integer m and n .

We will not take the semiclassical limit of the Green functions, but it is instructive to note that the Green function (6) has a semiclassical interpretation as a sum over all (non-diffractive) paths (j) from \mathbf{r} to \mathbf{r}'

$$G_G(\mathbf{r}, \mathbf{r}', E) \sim \sum_{J:q \rightarrow q'} G_{GV}^{(J)}(\mathbf{r}, \mathbf{r}', E) \quad (7)$$

where G_{GV} is given by the usual van Vleck–Gutzwiller expressions [4]. The symbolic transition $\dots j_i j_{i+1} \dots$ does not (in the semiclassical limit) correspond to *one* trajectory from \mathbf{r}_{j_i} to $\mathbf{r}_{j_{i+1}}$, as is usual in symbolic dynamics, but *all* trajectories from \mathbf{r}_{j_i} to $\mathbf{r}_{j_{i+1}}$.

Due to a singularity of the Hankel function $H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|)$, expression (6) diverges if $\mathbf{r} \rightarrow \mathbf{r}'$. We define a regularized geometric Green $\tilde{G}_G(\mathbf{r}, E)$ function by subtracting this singularity. The *diagonal* Green function from a disc to itself is then defined by

$$\tilde{G}_G(\mathbf{r} = 0, E) = -\frac{i}{4} \sum_{\rho \neq \mathbf{0}} H_0^{(1)}(k\rho) \quad (8)$$

and the *off diagonal* as before

$$\tilde{G}_G(\mathbf{r} \neq 0, E) = -\frac{i}{4} \sum_{\rho} H_0^{(1)}(k|\rho + \mathbf{r}|). \quad (9)$$

The multiplicative property of the weights t_p (3) and the existence of a complete symbolic dynamics enables the diffractive determinant, or zeta function, to be written in terms of a finite transfer matrix [22, 23]

$$T_{ij} = d(E) \cdot \tilde{G}_G(\mathbf{r}_j - \mathbf{r}_i, E) \quad (10)$$

via

$$\Delta_D(E) = \det(1 - \mathbf{T}). \quad (11)$$

2.2. Making the sums converge and the determinant real

The sums (9) and (8) diverge for real E and we will resort to the Ewald summation technique as developed in [25] in order to control the singularities. This procedure transforms the *diagonal* Green function to

$$\begin{aligned} \tilde{G}_G(0, E) &= \frac{1}{ab} \sum_{g=2\pi(m/a, n/a)} \frac{\exp(Q[1 - g^2/(2E)])}{2E - g^2} - \frac{1}{4\pi} \text{Ei}(Q) + \frac{i}{4} - \frac{1}{4\pi} \sum_{\rho \neq \mathbf{0}} I\left(\frac{k}{2}|\rho|\right) \\ &\equiv G^{(r)}(0, E) + \frac{i}{4} \end{aligned} \quad (12)$$

where $I(x)$ is defined by the integral

$$I(x) = \int_{\log(x/Q)}^{\infty} \exp(-2x \sinh \xi) d\xi. \tag{13}$$

These expressions are essentially identical to those in [25], we have just kept a and b as free parameters.

The off-diagonal Green function (9) is, after resummation,

$$\tilde{G}_G(\mathbf{r}_{12}, E) = \frac{1}{ab} \sum_{\mathbf{g}} \cos(\mathbf{r}_{12} \cdot \mathbf{g}) \frac{\exp(Q[1 - g^2/(2E)])}{2E - g^2} - \frac{1}{4\pi} \sum_{\rho} I\left(\frac{k}{2}|\rho + \mathbf{r}|\right) \tag{14}$$

$\mathbf{r}_{12} \neq 0.$

The derivation of this expression requires only a slight generalization of the derivation in [25], and so we omit it. Note that the off-diagonal terms are real.

In order to get a real expression for the determinant (11) we simply extract a factor $d(E)$ from each row

$$\Delta_D(E) = (-d)^n \det(\mathbf{M}). \tag{15}$$

The matrix elements of \mathbf{M}

$$M_{ij} = \begin{cases} \frac{1}{4} \frac{Y_0(kR)}{J_0(kR)} + \tilde{G}_G^{(r)}(0, E) & i = j \\ \tilde{G}_G(r_{ij}, E) & i \neq j \end{cases} \tag{16}$$

are then all real.

The energy dependence enters through the Green functions $\tilde{G}_G(\mathbf{r}, E)$ and the renormalized diffraction constant

$$\tilde{d} \equiv \frac{1}{4} \frac{Y_0(kR)}{J_0(kR)}. \tag{17}$$

We will in computations artificially fix \tilde{d} and keep the energy dependence only in the Green functions, for reasons that will be discussed later.

2.3. Numerical considerations

The numerical issue is to compute the Green functions with desired accuracy. The Ewald summation technique splits up the lattice sum into one sum over the dual lattice and one over the initial lattice $\sum I$. There is however no closed expression for the function $I(x)$, introduced in equation (13). The asymptotic behaviour of the function $I(x)$ is

$$I(x) = \frac{\exp(Q - x^2/Q)}{Q + x^2/Q} \left(1 - \frac{x^2/Q - Q}{(x^2/Q + Q)^2} \dots\right). \tag{18}$$

As Berry noted, for sufficiently small Q the sum $\sum I$ can be neglected, as far as the diagonal Green function is concerned. This is not so for the off-diagonal Green functions. They are still small in absolute terms, but are nevertheless significant. We have chosen to include the sum $\sum I$, and compute the integral (13) by the asymptotic formula when appropriate. However, for a small number of terms (a number decreasing with increasing energy) the integral has to be evaluated numerically. When $kr_{\min}/Q \gg 1$, where r_{\min} is the smallest inter-disc distance, this is no longer an issue. This suggests a small value of Q . On the other hand, a large Q is preferred in the dual lattice sum, so the choice of Q is a compromise and can be adjusted according to energy.

The determinant was derived in the limit of small kR . The first correction will involve the factor $Y_1(kR)/J_1(kR)$ so our diffractive determinant should work well whenever $kR \ll 1$. For

the $N = 1$ case the first correction involved the factor $Y_4(kR)/J_4(kR)$, the Bessel functions of order 1–3 are suppressed due to symmetry effects [25]. The diffractive approximation then works for slightly higher $kR \ll 4$. Indeed, we found in [16] that kR can be rather close to the limiting $kR = 4$ (for $N = 1$) and presumably rather close to $kR = 1$ in the general case ($N \geq 4$).

3. Mean level density

In this section we will focus on the mean density of zeros of the determinant $\Delta_D(E)$ as given in equations (11) and (15).

The density of states of the system is given asymptotically by Weyl's expression

$$\bar{d}_W = \frac{ab}{2\pi}. \quad (19)$$

The diffraction approximation underlying the derivation of the determinant $\Delta_D(E)$ does not ensure that it will reproduce the Weyl density. In fact, it will turn out that the average density of zeros of $\Delta_D(E)$ will depend on the number of scatterers according to

$$\bar{d}_D^{zeros} = \frac{\min(N, 4)}{4} \bar{d}_W. \quad (20)$$

That is, the full spectral density is achieved first when $N \geq 4$. We will study spectra for fixed values of the parameter $\tilde{d}(kR)$ for reasons to be discussed in the next section. But as the result (20) does not depend on the value of \tilde{d} , it will also apply to the (physical) case where $\tilde{d} = \tilde{d}(kR) = \tilde{d}(\sqrt{2ER})$ is allowed to vary with energy E .

Before actually deriving equation (20) we will make some general comments.

The reason why we do not resolve the full density of states for $N = 1$ and $N = 2$ has a simple explanation in terms of symmetries of the system. The wavefunctions split up into the irreducible representations of the respective group and our leading order determinant cannot resolve them all.

If $N = 1$ the symmetry is C_{2v} . In a coordinate system with its origin at the disc there is a reflection symmetry with respect to the x - and y -axis. We only resolve $\frac{1}{4}$ of the full spectral density, namely those states with even parity with respect to both axis. A small disc will be invisible if it sits at a point where the wavefunction, for symmetry reasons, is zero. To resolve the other subspaces one would need to take higher-order terms in the diffraction constant, and make a proper desymmetrization. If $N = 2$ there is an inversion symmetry with respect to the point $(r_1 + r_2)/2$ and we recover $\frac{1}{2}$ of the full spectral symmetry.

It is obvious that for high enough N one can avoid these kind of symmetry effects.

Now to the derivation of equation (20). First we show that the mean density of zeros of Δ_D equals the mean density of poles of the same determinant. To show this we can essentially repeat the arguments in [25]. Berry treated in that paper a slightly different determinant, but the basic mechanism is the same. The difference between the two integrated spectral densities is given by [25].

$$\begin{aligned} \bar{N}_D^{zeros}(E) - \bar{N}_D^{poles}(E) &= -\frac{1}{\pi} \langle \text{Im} \log \Delta_D(E + i\epsilon) \rangle = -\frac{1}{\pi} \langle \text{Im} \log \det(1 - T) \rangle \\ &= -\frac{1}{\pi} \langle \text{Im} \text{tr} \log(1 - T) \rangle = \frac{1}{\pi} \text{Im} \sum_{r=1}^{\infty} \frac{\langle \text{tr} T^r \rangle}{r}. \end{aligned} \quad (21)$$

A term $\text{tr} T^r$ is just a product of oscillating Hankel functions and is zero on the mean.

Next we will compute the density of poles of $\Delta_D(E)$. The poles of $\Delta_D(E)$ will be located at the poles of the Green functions whose density is $\bar{d}_W/4$. The problem is to determine their multiplicity m_N .

To summarize, we have so far:

$$\bar{d}_D^{\text{zeros}} = \bar{d}_D^{\text{poles}} = \frac{m_N}{4} \bar{d}_W. \quad (22)$$

We now study the behaviour of $\det(M)$ (with M defined in (16)), close to a pole corresponding to the quantum numbers $m, n > 0$, that is $2E$ is close to $g^2 = (2\pi)^2((m/a)^2 + (n/b)^2)$. The matrix elements are then (approximately) given by

$$M_{ij} = \delta_{i,j} \tilde{d} + \frac{4}{ab} \frac{\cos(2\pi m(x_j - x_i)) \cos(2\pi n(y_j - y_i))}{2E - 4\pi^2(m^2/a^2 + n^2/b^2)} \quad (23)$$

where we have summed over the four (dual) lattice points $2\pi(\pm m/a, \pm n/b)$. If we introduce the notation

$$\begin{aligned} \alpha_i &= 2\pi m x_i / a \\ \beta_i &= 2\pi n y_i / b \end{aligned} \quad (24)$$

and

$$\lambda = -\frac{ab\tilde{d}}{4} 2E - 4\pi^2(m^2/a^2 + n^2/b^2) \quad (25)$$

the latter measuring the (small) distance to the pole, we get

$$\det M = \left(\frac{\tilde{d}}{\lambda}\right)^N \det(\lambda \delta_{i,j} - \cos(\alpha_j - \alpha_i) \cos(\beta_j - \beta_i)) \equiv \left(\frac{\tilde{d}}{\lambda}\right)^N \det \tilde{M}. \quad (26)$$

The rank of the matrix \tilde{M} is simply the requested multiplicity m_N

$$\det M = \left(\frac{\tilde{d}}{\lambda}\right)^N O(\lambda^{N-\text{rank}(\tilde{M})}) \sim 1/\lambda^{\text{rank}(\tilde{M})}. \quad (27)$$

This rank is the maximum size a matrix having the structure

$$CC_{ij} = \cos(\xi_j - \xi_i) \cdot \cos(\eta_j - \eta_i) \quad (28)$$

can have with a non-vanishing determinant. To explore this problem we introduce three other matrices

$$\begin{aligned} CS_{ij} &= \cos(\xi_j - \xi_i) \cdot \sin(\eta_j - \eta_i) \\ SC_{ij} &= \sin(\xi_j - \xi_i) \cdot \cos(\eta_j - \eta_i) \\ SS_{ij} &= \sin(\xi_j - \xi_i) \cdot \sin(\eta_j - \eta_i). \end{aligned} \quad (29)$$

We further introduce the notation CC_j to mean the j th column of CC , and similarly for CS_j etc. The idea is now to write column CC_j as the following linear combination

$$\begin{aligned} CC_j &= \cos(\xi_j - \xi_{j-1}) \cos(\eta_j - \eta_{j-1}) CC_{j-1} - \cos(\xi_j - \xi_{j-1}) \sin(\eta_j - \eta_{j-1}) CS_{j-1} \\ &\quad - \sin(\xi_j - \xi_{j-1}) \cos(\eta_j - \eta_{j-1}) SC_{j-1} + \sin(\xi_j - \xi_{j-1}) \sin(\eta_j - \eta_{j-1}) SS_{j-1}. \end{aligned} \quad (30)$$

Similar relations hold for CS_j , SC_j and SS_j . The results can be conveniently expressed in terms of matrices:

$$\mathbf{u}_j = \mathbf{T}_j \mathbf{u}_{j-1} \quad (31)$$

where

$$\mathbf{u}_j = \begin{pmatrix} CC_j \\ CS_j \\ SC_j \\ SS_j \end{pmatrix} \quad (32)$$

and

$$\mathbf{T}_j = \mathbf{T}(\xi = \xi_j - \xi_{j-1}, \eta = \eta_j - \eta_{j-1}) \quad (33)$$

and

$$\mathbf{T}(\xi, \eta) = \begin{pmatrix} \cos(\xi) \cos(\eta) & -\cos(\xi) \sin(\eta) & -\sin(\xi) \cos(\eta) & \sin(\xi) \sin(\eta) \\ \cos(\xi) \sin(\eta) & \cos(\xi) \cos(\eta) & -\sin(\xi) \sin(\eta) & -\sin(\xi) \cos(\eta) \\ \sin(\xi) \cos(\eta) & -\sin(\xi) \sin(\eta) & \cos(\xi) \cos(\eta) & -\cos(\xi) \sin(\eta) \\ \sin(\xi) \sin(\eta) & \sin(\xi) \cos(\eta) & \cos(\xi) \sin(\eta) & \cos(\xi) \cos(\eta) \end{pmatrix}. \quad (34)$$

Please note that the elements of \mathbf{T} are scalars, the elements of \mathbf{u} are column vectors. The external index j has nothing to do with the internal structure of these objects. From the definition of \mathbf{T} we have the relation

$$\mathbf{T}(\xi_j - \xi_{j-1}, \eta_j - \eta_{j-1})\mathbf{T}(\xi_{j-1} - \xi_{j-2}, \eta_{j-1} - \eta_{j-2}) = \mathbf{T}(\xi_j - \xi_{j-2}, \eta_j - \eta_{j-2}). \quad (35)$$

The basic idea is now to explore whether it is possible to write the column vector $\mathbf{C}\mathbf{C}_n$ as a linear combination of the preceding columns $\mathbf{C}\mathbf{C}_j$ ($j < n$). We address the corresponding problem for $\mathbf{C}\mathbf{S}_n$, $\mathbf{S}\mathbf{C}_n$ and $\mathbf{S}\mathbf{S}_n$ simultaneously, and write

$$\mathbf{u}_n = \mathbf{T}_n \mathbf{u}_{n-1} = \mu_{n-1} \mathbf{u}_{n-1} + (\mathbf{T}_n - \mu_{n-1} \mathbf{E}) \mathbf{u}_{n-1} \quad (36)$$

where μ_{n-1} is a multiplier and \mathbf{E} is the unit matrix. We continue this procedure until we arrive at

$$\mathbf{u}_n = \sum_{j=2}^{n-1} \mu_j \mathbf{u}_j + \mathbf{S} \mathbf{u}_1 \quad (37)$$

where

$$\mathbf{S} = (\mathbf{T}_n \mathbf{T}_{n-1} \dots \mathbf{T}_2) - \mu_{n-1} (\mathbf{T}_{n-1} \mathbf{T}_{n-2} \dots \mathbf{T}_2) - \dots - \mu_2 (\mathbf{T}_2) \quad (38)$$

or

$$\mathbf{S} = \mathbf{T}(\xi_n - \xi_1, \eta_n - \eta_1) - \sum_{j=2}^{n-1} \mu_j \mathbf{T}(\xi_j - \xi_1, \eta_j - \eta_1). \quad (39)$$

So the first n columns of $\mathbf{C}\mathbf{C}$ are linearly dependent if and only if we can find multipliers such that

$$\mathbf{S}_{1j} = 0 \quad j \neq 1. \quad (40)$$

The number of multipliers are $n - 2$ and the number of equations to fulfil is three. So for generic parameters ξ_i and η_i the determinant of $\mathbf{C}\mathbf{C}$ is zero for $n \geq 5$. So $m_N = \text{rank } \tilde{\mathbf{M}} = \min(4, N)$. Together with equation (22) the announced result (20) follows.

The argument above applies to generic configurations of the scatterers. There might be members of the disordered ensemble (of measure zero) for which the argument above fails.

4. Level statistics

The discs are distributed randomly over the torus according to a uniform distribution. We compute spectra for individual members of this ensemble. We choose the lattice constants to be $a = 1$ and $b = 2^{1/4}$; exact degeneracies are avoided and the spectral statistics of the empty torus is perfectly Poissonian [27].

The critical parameter is $\tilde{d}(kR)$. As mentioned in the introduction, statistical studies suffer because of the finiteness of the sample. However, if R is sufficiently small, a sufficiently large sample can be obtained with essentially constant kR . Since we are restricted to the range

$kR < 1$ this would require such large values of k that it would be numerically intractable. Instead we artificially fix kR and compute the bottom part of the spectrum [16]. We compute around 600 levels for each configuration—computations do get a bit tedious for large N .

We are interested in two measures on the spectra, the integrated level spacing distribution $P(s) = \int_0^s p(s') ds'$ where $p(s)$ is the nearest-neighbour spacing distribution. Secondly, we investigate the two-point correlation function of levels

$$\begin{aligned}
 R(\epsilon) &= \left\langle \sum_{ij} \delta((E - E_i)\bar{d} + \epsilon/2) \cdot \delta((E - E_j)\bar{d} - \epsilon/2) \right\rangle_E \\
 &= \delta(\epsilon) + \left\langle \sum_{i \neq j} \delta((E - E_i)\bar{d} + \epsilon/2) \cdot \delta((E - E_j)\bar{d} - \epsilon/2) \right\rangle_E \\
 &\equiv \delta(\epsilon) + \tilde{R}(\epsilon)
 \end{aligned}
 \tag{41}$$

where the average is taken over a large number of energies. The correlation functions are computed over a gaussian window centred at the middle of the sample spectrum, its width about one sixth of the sample size. The results are then smeared with another Gaussian of width 0.2.

We learned from section 3 that the diffractive determinant first reproduces the full density of states when $N \geq 4$ so we will concentrate on those values of N . But, to connect to our results in [16] we will also include some results for $N = 1$. The reader should bear in mind that for $N = 1$ only a quarter of the full density of states is resolved and the reported result applies to this single subspace. Superposition of all four subspaces would result in more Poisson-like statistics.

The underlying spectrum of the empty torus reveals itself clearly in the spectrum for large values of \bar{d} . If $N = 1$ and $\bar{d} = \pm\infty$ the spectrum is Poissonian for a trivial reason; the zeros of the determinant have been pushed towards the poles of the Green function corresponding to the spectrum of the empty torus, cf [25]. If $N \geq 4$ and $\bar{d} \rightarrow \pm\infty$ four zeros will be pushed towards each pole. The corresponding limiting integrated level spacing distribution is then

$$P(s) = \frac{3}{4} + \frac{1}{4}(1 - \exp(-s/4)).
 \tag{42}$$

This limiting distribution is plotted in figure 1 together with results for two different values of \bar{d} ($N = 7$).

For $N = 1$ and $\bar{d} = 0$ the states are, so to say, repelled by the poles of the Green function which result in a spectrum exhibiting level repulsion. The level spacings distribution is very close to GOE, see [16] and figure 2. An exact agreement is not possible since the eigenvalues are locked between eigenvalues of the integrable torus. This locking is released for high enough N . The two-point correlation function $\tilde{R}(\epsilon)$ shows a clearer deviation from GOE than $P(s)$ for $N = 1$, see figure 4. One of the main questions is whether GOE can be approached as $N \rightarrow \infty$.

Increasing N only to $N = 4$, while keeping $\bar{d} = 0$ (corresponding to $kR = 0.899\dots$), yields exactly the opposite result. The level spacing distribution appears to be perfectly Poissonian, see figure 2. We find the result very surprising and have not been able to find any reasonable explanation. It is known that a Poisson-like distribution arises from independent superposition of spectra, so one could think that the determinant (for some unknown reason) factorizes. However, the reported distribution agrees better with the Poissonian prediction than with the statistics of four superposed Wigner spectra, see figure 3. So the statistics still appears to be Poissonian. One could also reply that $kR = 0.899\dots$ is too close to $kR \approx 1$ to be physically relevant. However, we know that for $N = 1$ the diffractive approximation is

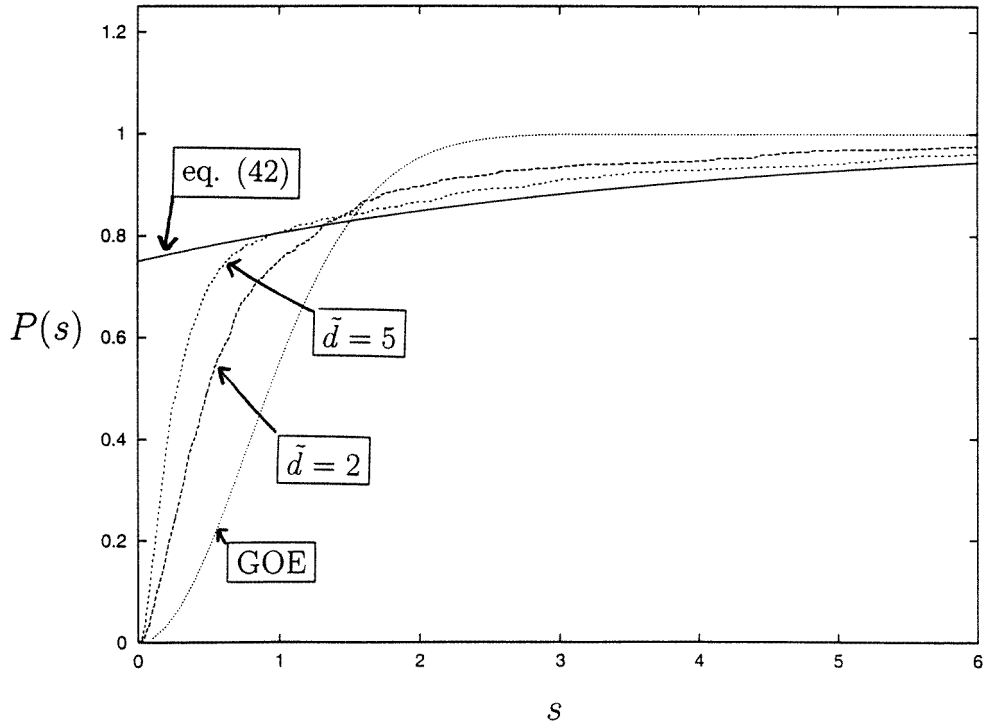


Figure 1. The integrated level spacings distribution for $N = 7$ for two different values of \tilde{d} .

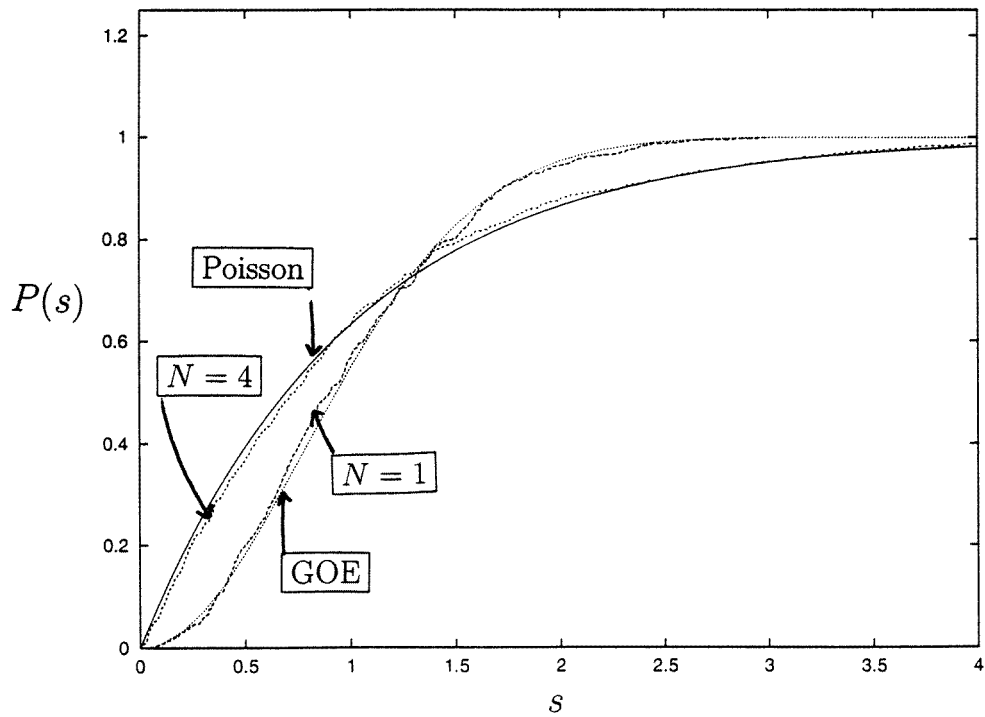


Figure 2. The integrated level spacings distribution for $\tilde{d} = 0$ for $N = 1$ and $N = 4$.

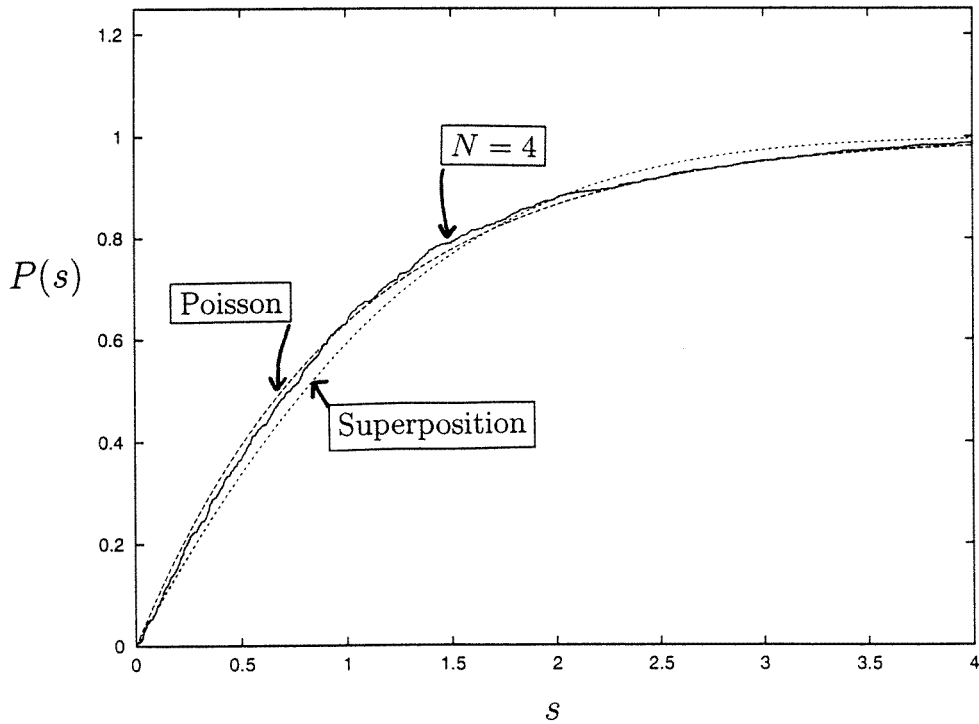


Figure 3. The integrated level spacings distribution for $\tilde{d} = 0$, $N = 4$ compared with the Poissonian spectrum and the result from four superposed Wigner spectra.

very good close to $kR = 0.899 \dots$ because a pole blows up the element in the KKR matrix that corresponds to the diffractive approximation, cf [16], the same thing should happen if $N = 4$.

In figure 4 we keep $\tilde{d} = 0$ constant and increase N further. From now on we restrict our investigations to the correlation function $\tilde{R}(\epsilon)$. According to the findings for $N = 1$ we expect it to be a better indicator of deviations from GOE.

We find that, indeed, the correlator seems to approach that of GOE; for $N = 17$ it already agrees much better with GOE than for $N = 1$.

This result is not restricted to any particular choices of kR , contrary to the $N = 1$ case. Next we are going to consider another series of data. Suppose we are increasing N and at the same time decreasing R in such a way that the fraction of the billiard area occupied by discs is kept constant: $NR_N^2 = C$. The corresponding spectra are then studied in the neighbourhood of some fixed k . For small values of kR we have [26]

$$\tilde{d} \equiv \frac{1}{4} \frac{Y_0(kR)}{J_0(kR)} \approx \frac{1}{2\pi} \left(\log \left(\frac{kR}{2} \right) + \gamma \right). \tag{43}$$

We choose arbitrarily $k = 2 \exp \gamma / \sqrt{C}$ and thus $\tilde{d} \approx -\log N / 4\pi$, and we are led to study the sequence

$$\tilde{d}_N = -\frac{\log N}{4\pi}. \tag{44}$$

The trend is the same, see figure 5. The correlation function approaches the prediction of GOE, but the approach is of course much slower.

The conclusions suggested by these studies are summarized in table 1. The result in the lower right-hand corner applies if the limit is approached according to equation (44).

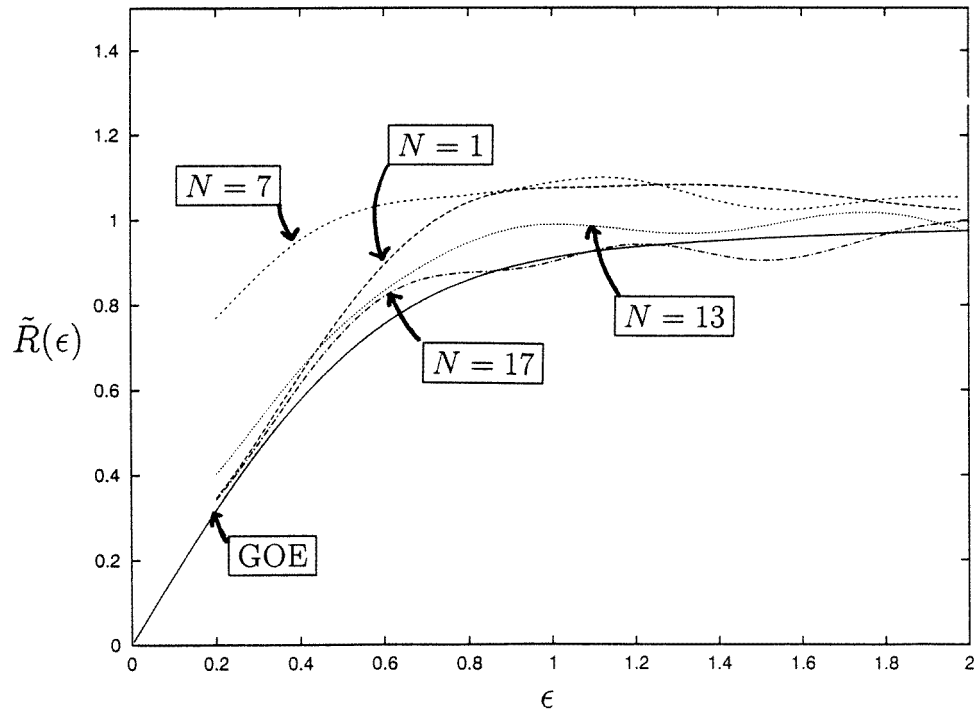


Figure 4. The correlation function $\tilde{R}(\epsilon)$ for $\tilde{d} = 0$ for variable number of scatterers, compared with the GOE prediction.

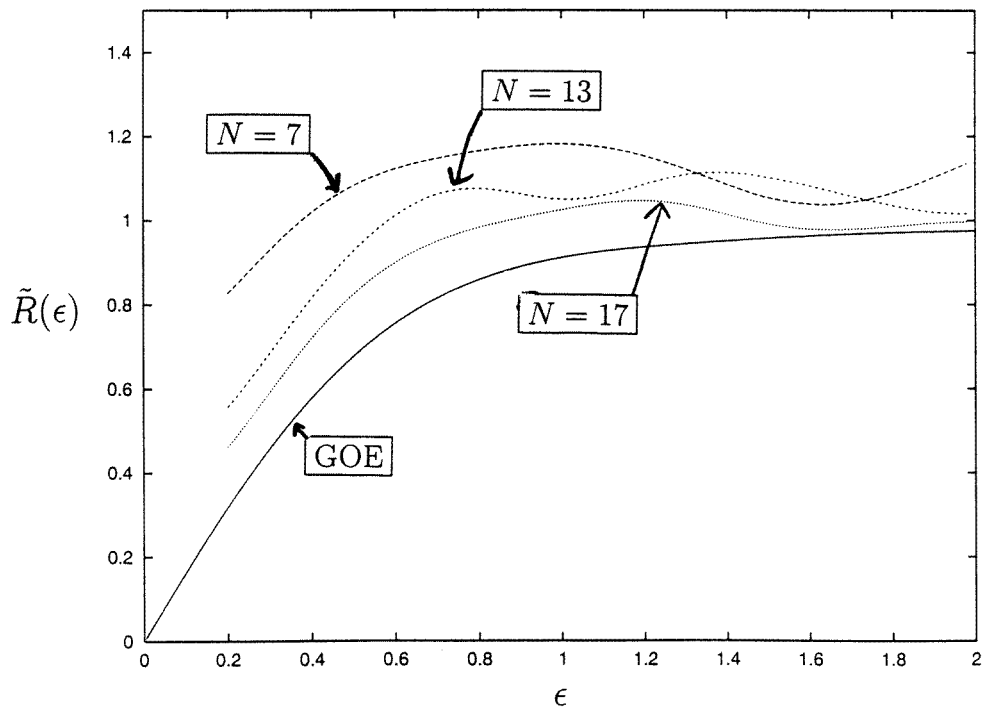


Figure 5. The correlation function $\tilde{R}(\epsilon)$ for \tilde{d} chosen according to equation (44), for variable number of scatterers, compared with the GOE prediction.

Table 1. Suggested level statistics for various values of the parameters N and \tilde{d} (kR).

	$ \tilde{d} = 0$	$0 < \tilde{d} < \infty$	$ \tilde{d} = \infty$
$N = 1$	\approx GOE	—	Poisson
$N = 4$	Poisson	—	$4 \times$ Poisson
$4 < N < \infty$	—	—	$4 \times$ Poisson
$N = \infty$	GOE	GOE	GOE

5. Discussion

The emergence of GOE in the limit of many small scatterers hardly comes as a big surprise. We consider the method rather than the results interesting—the results have been obtained within the framework of periodic orbit theory.

Our numerical investigations of the systems are by no means exhaustive, we have merely scratched the surface. We would like to perform further work on this class of system including the following.

- Improve statistics.
- The system is so simple that an analytical treatment does not seem entirely out of reach.
- For regular lattices the diffractive determinant factorizes into a product over Bloch states. The band structure of these systems will be addressed in a forthcoming paper.
- Study the transition from a regular structure to a disordered one.
- We also plan to study conductance by means of the Landauer formula [28, 29]. The simple composition of Green functions of our diffractive system is perfectly suited for the Landauer formula, which can be formulated in terms of transfer matrices.

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